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A disorder parameter for dual superconductivity in gauge theories.

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A detailed discussion is given of the disorder parameter for dual superconductivity of lattice gauge theories, introduced in a previous paper, and of its relation to other approaches. New lattice data are reported. Among other results, we find that the dual superconductivity of compact $U(1)$ is type II.

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I. INTRODUCTION

Dual superconductivity of the vacuum is an important phenomenon in gauge theories.

- It produces confinement of electric charges via dual Meissner effect in the abelian case.
- It is most likely the mechanism of colour confinement in *QCD* [1,2]
- It plays a key role in the structure of supersymmetric gauge theories [3].

The simplest case is the compact $U(1)$ gauge theory.

With Wilson action this theory shows a phase transition at $\beta_c \simeq 1.01$, probably weak first order, from a phase at low β where electric charge is confined, to a phase of free photons [4].

Confinement is detected by measuring the string tension from the vacuum expectation value (*vev*) of Wilson loops. The penetration depth of the electric field is finite for $\beta < \beta_c$, indicating dual Meissner effect, and goes large at the deconfining transition β_c [4].

Monopoles are detected by their Dirac strings as units of 2π magnetic flux through the plaquettes. Their number density is not a disorder parameter for dual superconductivity, in the same way as the number of electric charges is not for ordinary superconductivity. However, empirically, the number density of monopoles is larger in the confined phase, and drops to zero above β_c [4].

A legitimate disorder parameter should vanish for symmetry reasons in the deconfined phase, and be different from zero in the confining phase. Since dual superconductivity is nothing but the spontaneous breaking of the $U(1)$ symmetry related to the magnetic charge conservation, the *vev* of any operator carrying non zero magnetic charge can be a disorder parameter.A non zero *vev* of such an operator would indeed indicate that vacuum has not a definite magnetic charge, i.e. that monopoles condense in it in the same way as Cooper pairs do in the ground state of ordinary superconductors. The concept of disorder parameter is known since long time in the comunity of field theory and statistical mechanics [5,6]. In the pionering numerical simulations in lattice gauge theories, however, emphasis was given to the density of monopoles as indicators of dual superconductivity [4,7].

A rigorous proof was given in ref. [8] that monopoles condense at low β's in lattice $U(1)$ theory with Villain action [9]. The proof makes use of the specific form of the action and so did numerical attempts to extract a disorder parameter [10]. However, probably because of the mathematical language of the forms, which is not so familiar to physicists, nobody tried for long time to export the construction to the generic form of the action, or to non abelian gauge theories. Indeed, after abelian projection [11], monopole condensation in non abelian gauge theories like *QCD*, always reduces to an effective $U(1)$ with Dirac monopoles [11,12]. Of course the $U(1)$ effective action is unknown, and therefore a construction of the disorder parameter is needed, which can work with any variant action.

Such a construction was given in ref [13] and immediately afterwards was used to demonstrate dual superconductivity of non abelian theories [14].

This result prompted the exportation of the construction of ref. [8,10] from Villain to generic action.

In this paper we want to discuss in detail and improve the construction of ref [13] (sect.2), compare it to that of ref. [8], (sect.3), showing that they are equivalent, and present a number of numerical results for lattice $U(1)$ with Wilson's action, (sect.4). We will then compare our [13] way of detecting superconductivity by the quantity $\rho = \frac{d}{d\beta}\ln\langle\mu\rangle$, $\langle\mu\rangle$ being the disorder parameter, to direct determination of $\langle\mu\rangle$ or of its effective potential (sect.4).

Besides confirming dual superconductivity of $U(1)$ gauge theory in the confined phase we show that it is

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II kind.

The present discussion is also a useful basis to the treatment of the analogous problem in non abelian theories, which will be presented elsewhere.

II. THE DISORDER PARAMETER.

The construction of ref [13] of the creation operator of a monopole or antimonopole, is inspired by ref [5,6] and is based on the following simple idea.

In the Schrödinger representation where the field $\vec{A}(x)$ is diagonal, a monopole of charge $2\pi\frac{q}{e}$ sitting in $\vec{y}$ is created by adding the corresponding vector potential $\frac{1}{e}\vec{b}(\vec{x}-\vec{y})$ to $\vec{A}(x)$.

This is nothing but a translation of $\vec{A}(x)$, which is generated by the conjugate momentum $\vec{\pi}(x) = \vec{E}(x)$, the electric field operator. In the same way as

$$\mathrm{e}^{\mathrm{i}pa}|x\rangle = |x+a\rangle \tag{1}$$

we have

$$|\vec{A}(\vec{x},t) + \frac{1}{e}\vec{b}(\vec{x}-\vec{y})\rangle = \mu|\vec{A}(\vec{x},t)\rangle \tag{2}$$

with

$$\mu(\vec{y},t) = \exp\left[\mathrm{i}\frac{1}{e}\int \mathrm{d}^3x\, \vec{E}(\vec{x},t)\vec{b}(\vec{x}-\vec{y})\right] \tag{3}$$

The magnetic charge operator being

$$Q = \int \mathrm{d}^3x \vec{\nabla}\left(\vec{\nabla}\wedge\vec{A}(\vec{x},t)\right) \tag{4}$$

the commutator $[Q,\mu]$ can be evaluated by use of the canonical commutation relations

$$[E_i(\vec{x},t), A_j(\vec{z},t)] = -\mathrm{i}\delta_{ij}\delta^3(\vec{x}-\vec{z}) \tag{5}$$

giving

$$\begin{aligned}[Q(t),\mu(\vec{y},t)] &= \frac{1}{e}\int \mathrm{d}^3x\vec{\nabla}\left(\vec{\nabla}\wedge\vec{b}(\vec{x}-\vec{y})\right)\cdot\mu(\vec{y},t)\\ &= \frac{q}{2e}\mu(\vec{y},t)\int \mathrm{d}^3x\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) =\\ &= 2\pi\frac{q}{e}\mu(\vec{y},t)\end{aligned} \tag{6}$$

In deriving Eq.(6) the Dirac string has been removed.

A choice for $\vec{b}(\vec{x}-\vec{y})$ can be

$$\vec{b}(\vec{x}-\vec{y}) = \frac{q}{2}\frac{\vec{r}\wedge\vec{n}_3}{r(r-\vec{r}\cdot\vec{n}_3)} \tag{7}$$

Alternative choices differ by a gauge transformation, $\vec{b}\to\vec{b}+\vec{\nabla}\Phi$ which leaves the operator invariant if the Gauss law $\vec{\nabla}\cdot\vec{E}=0$ is satisfied.

On the lattice the building block of the theory is the link $U_\mu(n)$, which is an element of the gauge group. For $U(1)$ $U_\mu(n) = \mathrm{e}^{\mathrm{i}\theta_\mu(n)}$ and the plaquette, $\Pi_{\mu\nu}$, which is the parallel transport along the elementary square in the plane $\mu\nu$ at the site n, is

$$\Pi_{\mu\nu}(n) = \exp(\mathrm{i}\theta_{\mu\nu}(n)) \tag{8}$$

with

$$\theta_{\mu\nu} = \Delta_\mu\theta_\nu - \Delta_\nu\theta_\mu \underset{a\to 0}{\simeq} a^2 e F_{\mu\nu} \tag{9}$$

The lattice version of the electric field is then

$$a^2 E_i \simeq \frac{1}{e}\mathrm{Im}\,\Pi^{0i} + \mathcal{O}(a^4) \tag{10}$$

and a definition of the operator μ on the lattice [13] can be

$$\begin{aligned}\mu(\vec{y},n_0) &= \exp\left[-\beta\sum_n b^i(\vec{n}-\vec{y})\mathrm{Im}\Pi^{0i}(\vec{n},n_0)\right] =\\ &= \exp\left[-\beta\sum_{\vec{n}} b^i(\vec{n}-\vec{y})\sin(\theta_{0i}(\vec{n},n_0))\right]\end{aligned} \tag{11}$$

$\beta = 1/e^2$. Here $b^i(\vec{n})$ is the discretized version of the monopole field Eq.(7). The factor β in front of the exponent comes from the factor $\frac{1}{e}$ in the monopole charge times the normalization factor in Eq.(10). Usual Wick rotation to Euclidean region has been performed.

The form (11) was successfully used in ref. [13].

A better definition of μ can be given, which coincides with Eq.(11) in practice, but automatically respects the compactness of the theory, in that it shifts the exponent of the links, and not the links themselves. In formulae:

$$\begin{aligned}\mu(\vec{y},m_0) = \exp\Big\{\beta\sum_{\vec{n}}\big(&\cos[\theta^{0i}(\vec{n},m_0)+b^i(\vec{y}-\vec{n})] -\\ &-\cos[\theta^{0i}(\vec{n},m_0)]\big)\Big\}\end{aligned} \tag{12}$$

For small b^i the definition (12) coincides with (11).

More generally if $\sum_{\mu\nu n} S(\theta_{\mu\nu}(n))$ is the action, μ will be defined as

$$\begin{aligned}\mu(\vec{y},m_0) = \exp\Big\{\beta\sum_{\vec{n}}\big[&S(\theta^{0i}(\vec{n},m_0)+b_i(\vec{n}-\vec{y})) -\\ &- S(\theta^{0i}(\vec{n},m_0))\big]\Big\}\end{aligned} \tag{13}$$

and will tend to the expression (11) as the lattice spacing a go to zero, when the action tends to the continuum action.

The prescription of excluding Dirac string on a lattice being either to locate the monopole at $\vec{y}$ between two neighbouring sites, or to eliminate in the sum the arrow of sites where $\vec{b}$ is singular, it is easy to verify that the definitions (12) and (11) give the same results from the practical point of view.

If the action is the Wilson's action [15]

$$S = \sum_{n,(\mu\nu)} \beta\left(\cos(\theta_{\mu\nu}) - 1\right) \qquad (14)$$

then the vacuum expectation value of μ is given by

$$\langle\mu\rangle = \frac{1}{Z}\int \left[\prod_{\mu,n} \mathrm{d}\theta_\mu(n)\right] \exp(S)\,\mu \qquad (15)$$

or, making use of (12)

$$\langle\mu(\vec{y},m_0)\rangle = \frac{1}{Z}\int \left[\prod_{\mu,n} \mathrm{d}\theta_\mu(n)\right] \exp(S+S') \qquad (16)$$

where S' is the exponent of Eq.(12).

Adding S' simply amounts to modify the $(0,i)$ plaquettes on the time slice n_0, by addition of b_i to θ_{0i}

$$\begin{aligned} S+S' = &\sum_n \sum_{(i,j)=1}^{3} \beta\left(\cos(\theta_{ij}(n) - 1\right) + \qquad (17)\\ &+ \sum_{n,n_0\neq m_0} \beta\left(\cos(\theta_{0i}(n) - 1\right) + \\ &+ \sum_{\vec{n}} \left(\cos(\theta_{0i}(\vec{n},m_0) + b^i(\vec{m}-\vec{n})) - 1\right) \end{aligned}$$

If a number of monopoles and antimonopoles are created at time n_0, b_i should be the sum of the corresponding vector potentials. The generic correlation function $\langle\mu(x_1)\ldots\mu(x_n)\rangle$ is defined as $\langle\mu\rangle$ in Eq.(16), with the change from S to $S+S'$ extended to all the time slices where monopoles or antimonopoles are created.

So for example the correlation function where a monopole is created in $\vec{y}=0$ at $t=0$ and destroyed at $t=n_0$ is given by

$$\langle\mu(\vec{y},0)\,\bar{\mu}(\vec{y},m_0)\rangle = \frac{1}{Z}\int \exp(S+S'_{\mu\bar{\mu}})$$

$S+S'$ differs from S by the replacement

$$\begin{aligned} \theta_{0i}(\vec{n},0) &\to \theta_{0i}(\vec{n},0) + b_i(\vec{n}-\vec{y}) \qquad \text{at}\, t=0 \\ \theta_{0i}(\vec{n},m_0) &\to \theta_{0i}(\vec{n},m_0) - b_i(\vec{n}-\vec{y}) \qquad \text{at}\, t=m_0 \qquad (18) \end{aligned}$$

Monopole condensation can be detected from the asymptotic value of $\langle\mu(\vec{y},0)\,\bar{\mu}(\vec{y},n_0)\rangle$. Indeed as n_0 grows large, by cluster property

$$\langle\mu(\vec{y},0)\,\bar{\mu}(\vec{y},m_0)\rangle \simeq C\,\exp(-m_0 M) + \langle\mu\rangle^2 \qquad (19)$$

Notice that $\langle\mu\rangle = \langle\bar{\mu}\rangle$ by C invariance, and the position $\vec{y}$ is irrelevant by translation invariance. M is the mass of the lowest state with monopole charge q in units of inverse lattice spacing.

To visualize that μ really creates a monopole at $t=0$ consider again the change it produces according to Eq.(18). Since

$$\theta_{0i}(\vec{n},0) = \theta_i(\vec{n},1) - \theta_0(\vec{n},0) - \theta_0(\vec{n}+\hat{i},0) + \theta_0(\vec{n},0) \qquad (20)$$

the change (18) of θ_{0i} can be considered as a shift

$$\theta_i(\vec{n},1) \to \theta_i(\vec{n},1) - b_i(\vec{n}-\vec{y}) \qquad (21)$$

A change of variables

$$\theta'_i = \theta_i(\vec{n},1) - b_i(\vec{n}-\vec{y}) \qquad (22)$$

in the Feynman integral (16), which leaves the measure invariant, brings back the plaquette θ_{0i} to its unperturbed form. However the change of variables (22) changes the (i,j) plaquette at $n_0=1$ as follows

$$\theta_{ij}(\vec{n},1) \to \theta_{ij}(\vec{n},1) + \Delta_i b_j(\vec{n}-\vec{y}) - \Delta_j b_i(\vec{n}-\vec{y}) \qquad (23)$$

This means that at $n_0 = 1$ the magnetic field of a monopole located at $\vec{n}=\vec{y}$ is added to the original configuration. The change of variables (21) also affects the plaquette $\theta_{0i}(\vec{n},2)$, and amounts to the shift

$$\theta_{0i}(\vec{n},2) \to \theta_{0i}(\vec{n},2) - b_i(\vec{n}-\vec{y}) \qquad (24)$$

Again a change of variables $\theta_i(\vec{n},2) \to \theta_i(\vec{n},2) - b_i(\vec{n}-\vec{y})$ restores $\theta_{0i}(\vec{n},2)$ to the initial form at the price of adding a monopole at time $t=2$, and of producing a shift in the form (24) on $\theta_{0i}(\vec{n},3)$. This procedure can be iterated. At $t=m_0$ this procedure ends, because b_i cancels with the shift of opposite sign corresponding to the creation of the antimonopole.

Thus the correlator $\langle\mu(\vec{y},0)\,\bar{\mu}(\vec{y},m_0)\rangle$ simply consists in having a monopole propagating in time, from 0 to n_0.

The construction above simply generalizes to more complicated forms of the action, where Wilson loops other than plaquettes enter.

III. COMPARISON WITH OTHER APPROACHES

In this section we want to discuss the relation of our approach to that of ref. [8].

In the language of ref. [8] $\theta_\mu(n)$ is a 1 form associated to the links and $\mathrm{d}\,\theta$ is the two form associated to the plaquettes, or the field strength tensor.

In this language the partition function is

$$Z = \int \mathcal{D}[\theta]\,\Phi_\beta(\mathrm{d}\theta) \qquad (25)$$

For Wilson's action

$$\Phi_\beta = \exp(\beta \sum_{plaq} (\cos(\mathrm{d}\theta) - 1) \qquad (26)$$

For Villain's action

$$\Phi_\beta = \sum_n \exp\left\{-\frac{\beta}{2}\sum_{plaq} \|\mathrm{d}\theta + 2\pi n\|^2\right\} \tag{27}$$

To define a disorder operator $\langle\mu\rangle$ the action is modified by adding a two form X to $\mathrm{d}\theta$. We define

$$Z(X) = \int \mathcal{D}[\theta]\, \Phi_\beta(\mathrm{d}\theta + X) \tag{28}$$

and

$$\langle\mu\rangle = \frac{Z(X)}{Z(0)} \tag{29}$$

Any change of X of the form $X \to X + \mathrm{d}\Lambda$ leaves $\langle\mu\rangle$ invariant, in that $\mathrm{d}\Lambda$ corresponds to a shift of θ to $\theta + \Lambda$ which is reabsorbed by a change of the (periodic) integration variables.

Since a generic X can be written as (Hodge decomposition):

$$X = \mathrm{d}\alpha + \delta\frac{1}{\Delta}\mathrm{d}X \tag{30}$$

the above invariance implies that $\langle\mu\rangle$ only depends on $\mathrm{d}X$.

$\mathrm{d}X$ is a 3-form $[\mathrm{d}X]_{\mu\nu\alpha}$ and its dual $*\mathrm{d}X$ is a 1 form, which is a magnetic current, since X is a field strength. Explicitely

$$\mathrm{d}X_{\mu\nu\alpha} = -\left(\partial_\alpha X_{\mu\nu} + \partial_\mu X_{\nu\alpha} + \partial_\nu X_{\alpha\mu}\right) \tag{31}$$

and

$$J^M_\rho = \frac{1}{6}\varepsilon_{\rho\mu\nu\alpha}\,\mathrm{d}X_{\mu\nu\alpha} \tag{32}$$

The magnetic current (32) is identically conserved. In the language of forms

$$\delta J^M = 0 \tag{33}$$

The magnetic charge density which describes the creation of a monopole of charge $2\pi q$ in the site $\vec{y}$ at time y^0, and its destruction at time y'^0 is

$$J^M_0(\vec{x}, x^0) = 2\pi q\delta^3(\vec{x} - \vec{y})\left(\theta(x^0 - y^0) - \theta(x^0 - y'^0)\right) \tag{34}$$

Since the current is conserved

$$\begin{aligned}\vec{\nabla}\vec{J}^M &= -\Delta_0 J^M_0 = \\ &= -2\pi q\delta^3(\vec{x} - \vec{y})\left(\delta(x^0 - y^0) - \delta(x^0 - y'^0)\right)\end{aligned} \tag{35}$$

A solution of Eq.(35) is

$$\vec{J}^M(\vec{x}, x^0) = 2\pi q\frac{1}{4\pi}\frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}\left[\delta(x^0 - y^0) - \delta(x^0 - y'^0)\right] \tag{36}$$

The corresponding X is then

$$\overline{X} = \delta\frac{1}{\Delta}J^M \tag{37}$$

The correlation function of a monopole antimonopole will then be:

$$\langle\mu(\vec{y}, y_0)\,\overline{\mu}(\vec{y}, y'^0)\rangle = \frac{Z(\overline{X})}{Z(0)} \tag{38}$$

This is the construction of ref. [8].

Notice that $Z(X)$ is periodic in X (with period 2π) since the action is compact. In fact Z only depend on $\mathrm{d}X$, and is periodic also in $\mathrm{d}X$ with the same period. This can be rigorously proved by going to Fourier transform:

$$Z(\mathrm{d}X + 2\pi n) = Z(\mathrm{d}X) \tag{39}$$

Consider now a one form Ω on the dual lattice, with support on a line. If $\delta\Omega = 0$ the support must be a closed line. If Ω is integer valued in units of 2π the change

$$\mathrm{d}X = *J^M \to \mathrm{d}X = *J^M + \Omega$$

leaves Z invariant.

In the notation of ref. [8] $\vec{J}^M$ is denoted by $2\pi qB$ and J^M_0 by $-2\pi q\omega$ and

$$\mathrm{d}\overline{X} = 2\pi q(B - \omega) \tag{40}$$

Any X with the same $\mathrm{d}X$, will give the same correlation function (38). The construction presented in sect. 2 corresponds to the choice

$$\begin{aligned}\overline{X}'_{0i} &= b_i(\vec{x})\left[\delta(x^0 - y^0) - \delta(x^0 - y'^0)\right] \\ \overline{X}'_{ij} &= 0\end{aligned} \tag{41}$$

or, in the dual language

$$\begin{aligned}&(*\overline{X}')_{0i} = 0 \\ &(*\overline{X}')_{ij} = \varepsilon_{ijk}b_k(\vec{x})\left[\delta(x^0 - y^0) - \delta(x^0 - y'^0)\right]\end{aligned} \tag{42}$$

and

$$*\,\mathrm{d}\overline{X}'_\mu = \delta(*\overline{X}')_\mu = -\sum_\rho \Delta_\rho(*X)_{\rho\mu} \tag{43}$$

Explicitely

$$\delta(*\overline{X}')_0 = 0 \qquad \delta(*\overline{X}')_i = -\sum_k \Delta_k(*X)_{ki}$$

and by Eq.'s (41) and (7)

$$\begin{aligned}\delta(*\overline{X}')_i = {} & 2\pi q\frac{1}{4\pi}\frac{x_j - y_j}{|\vec{x} - \vec{y}|^3}\left(\delta(x_0 - y_0) - \delta(x_0 - y'_0)\right) \\ & - 2\pi q\delta(x_1 - y_1)\delta(x_2 - y_2)\theta(x_3 - y_3)\,\cdot \\ & \cdot\,\left(\delta(x_0 - y_0) - \delta(x_0 - y'_0)\right)\end{aligned} \tag{44}$$

Our $*\mathrm{d}\overline{X}'$ differs from $*\mathrm{d}\overline{X}$ (40) of ref. [8] by a 1 form integer valued in units of 2π, with support on a closed line. Therefore our correlator coincides with that of ref. [8], not only for Villain action, but for generic form of the action.

This section is a cultivated way of presenting the argument already given at the end of last section.

IV. NUMERICAL RESULTS FOR THE DISORDER PARAMETER.

As discussed in sect. 2, we measure the correlation function

$$\mathcal{D}(x^0) = \langle \mu(\vec{x}, x^0), \bar{\mu}(\vec{x}, 0) \rangle \simeq A\mathrm{e}^{-Mx^0} + \langle \mu \rangle^2 \qquad (45)$$

The aim is to extract $\langle \mu \rangle^2$, which will signal dual superconductivity, and M which is the lowest mass in the sector of magnetically charged excitations.

A direct determination of $\mathcal{D}$ can be done, as we will discuss below, but is rather noisy from numerical point of view. The reason for this is that $\mathcal{D}$

$$\mathcal{D} = \frac{1}{Z} \int \mathcal{D}\theta \, \exp(S + S') \qquad (46)$$

is the average of $\exp(S')$, S' being the modification of the action on the time slices $t = 0$ and $t = x^0$, and S' fluctuates roughly like the square root of the spatial volume.

A way to go around this difficulty is to measure, instead of $\mathcal{D}$ the quantity [12]

$$\rho(\vec{x}, x^0, \vec{x}, 0) = \frac{d}{d\beta} \ln \mathcal{D} \qquad (47)$$

At large distance ($x^0 \to \infty$)

$$\rho_\infty \simeq 2 \frac{d}{d\beta} \ln \langle \mu \rangle \qquad (48)$$

and since $\rho(\beta = 0) = 1$ $\langle \mu \rangle$ can be reconstructed as

$$\langle \mu \rangle = \exp\left(\frac{1}{2} \int \rho(\beta') d\beta' \right) \qquad (49)$$

From Eq.(46)

$$\rho_\infty = \langle S \rangle_S - \langle S + S' \rangle_{S+S'} \qquad (50)$$

The definition of ρ is analogous to the definition of the internal energy in terms of the partition function in statistical mechanics. ρ is now a well defined quantity and easy to measure, and, as we shall see, can give all the information needed to detect dual superconductivity.

We have made simulations on a $6^3 \times 12$, $8^3 \times 16$ and $10^3 \times 20$ lattices putting the time axis along the long edge of the lattice. A typical behaviour of ρ versus x^0 is shown in Fig.1, for a $8^3 \times 16$ lattice, showing that an asymptotic value is reached by ρ as a function of x^0. The mass M of the exponential in Eq.(45) can be estimated and is typically $\sim (2-3)/a$

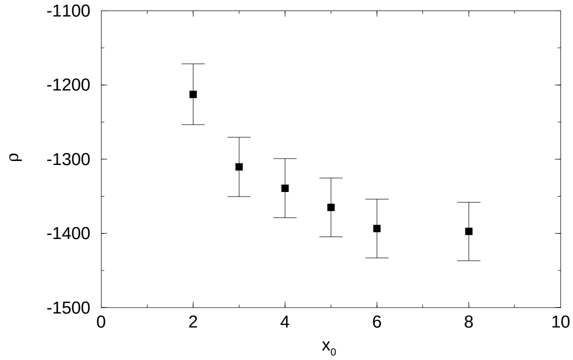

Fig.1 Monopole antimonopole correlation in time. (Lattice $8^3 \times 16$)

We will come back again to this point in the following. The quantity ρ_∞ as a function of β is plotted in Fig.2.

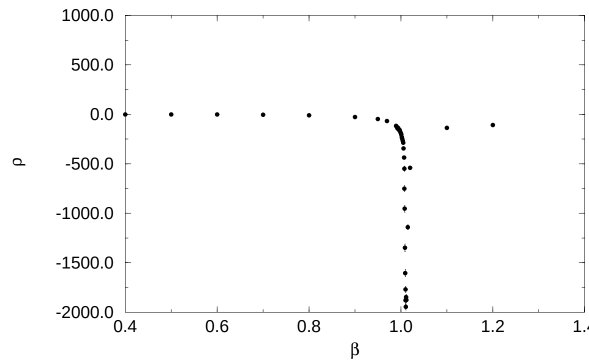

Fig.2 ρ_∞ as a function of β. The negative peak signals the phase transition. (Lattice $8^3 \times 16$)

For all of our lattices sizes ρ_∞ is negative and sharply decreases approaching β_c. This corresponds, by Eq.(49) to a behaviour of $\langle \mu \rangle$ which slowly decreases from the value $\langle \mu \rangle = 1$ at $\beta = 0$, and has a sharp drop at β_c.

To better analyse this behaviour we compare it for the three lattice sizes under study. For $\beta < \beta_c$ below the negative peak, ρ increases with L, showing that as $L \to \infty$, $\langle \mu \rangle$ reaches a finite, nonzero value. Magnetic $U(1)$ is therefore spontaneously broken, and for $\beta < \beta_c$ the system is a dual superconductor (fig.3).

For $\beta \simeq \beta_c$ we know that the typical correlation length of the system goes large. There is evidence that the transition is weak first order [16], with some controversy [17].

The correlation length ξ goes large as β approaches β_c in a range of β's and eventually stops growing before reaching it.

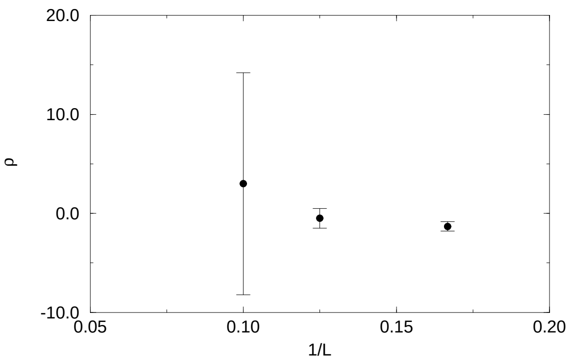

Fig.3 ρ_∞ versus $1/L$ for $\beta 1.009$.

This means that, in the neighbourhood of β_c

$$\mu = \mu\left(\frac{\xi}{L}, \frac{a}{\xi}\right) \simeq \mu\left(\frac{\xi}{L}\right) \tag{51}$$

If the transition were second order a critical index ν would exist such that

$$\xi \underset{\beta \to \beta_c^-}{\simeq} (\beta_c - \beta)^{-\nu} \tag{52}$$

In our case some effective index ν could anyhow exist, describing a behaviour of ξ of the form (52) in the above mentioned range of β's. Then ξ/L can be traded with $L^{1/\nu}(\beta_c - \beta)$ and a finite size scaling behaviour results

$$\mu = \mu[L^{1/\nu}(\beta_c - \beta)] \tag{53}$$

implying for $\rho = \frac{d}{d\beta}\ln\langle\mu\rangle$ a scaling behaviour

$$\frac{\rho}{L^{1/\nu}} = f\left(L^{1/\nu}(\beta_c - \beta)\right) \tag{54}$$

Eq.(54) allows a determination of ν and β_c, togheter with a determination of the exponent δ by which $\langle\mu\rangle$ tends to zero at β_c in the infinite volume limit.

The quality of the scaling is shown in Fig.4. points corresponding to different lattice sizes follow the same universal curve only for the appropriate values of β_c and ν, Eq.(55). If β_c or ν are changed by one standard deviation from the values of Eq.(55) points from different lattices start splitting apart from each other.

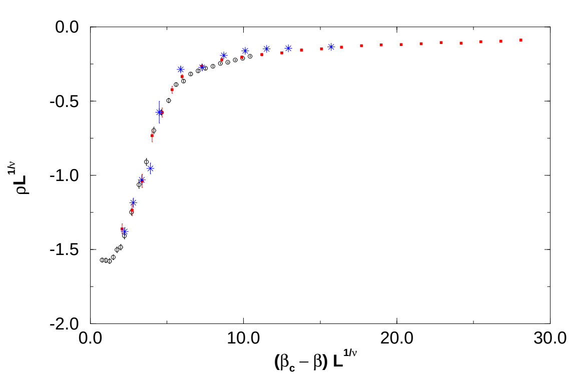

Fig.4 Finite size scaling. $\rho L^{1/\nu}$ is plotted vs. $(\beta_c - \beta)L^{1/\nu}$.

A best square fit gives

$$\beta_c = 1.01160(5) \qquad \nu = 0.29(2) \tag{55}$$

The value (55) of β_c is consistent with determination based on completely different methods [16]. If $\mu \to (\beta_c - \beta)^\delta$

$$\frac{\rho}{L^{1/\nu}} \simeq -\frac{\delta}{L^{1/\nu}(\beta_c - \beta)} \tag{56}$$

An estimate for δ from the behaviour in fig.4 is:

$$\delta = 1.1 \pm 0.2 \tag{57}$$

In the region $\beta \to \infty$ ρ can be computed in the weak coupling approximation [13]. The result is

$$\rho = -5.05 \cdot L + 4.771 \tag{58}$$

giving $\rho \to -\infty$ or $\langle\mu\rangle = 0$ in the infinite volume limit, in agreement with general arguments [5]: only as $V \to \infty$ the disorder parameter vanishes in the disordered phase, if boundary conditions are not free.

The mass of the monopole in Eq.(45) should scale properly in the limit $\beta \to \beta_c$ but we have large errors and this behaviour is not clearly visible (fig.5).

In order to determine if the superconductor is first kind or second kind we have also measured the penetration depth $1/m_A$ of the electric field on the lines of ref. [4][1]. A constant electric field parallel to the space boundary of the lattice is put on a face of the space lattice and its value is determined inside the bulk as a function of the distance from the boundary. An exponential behaviour is found, with a penetration depth which properly scales by approaching the critical point, consistently with the effective critical index.

The corresponding mass is shown in fig.5 together with the mass extracted from the correlation length, Eq.(45). It appears clearly that $M \geq 2m_A$, indicating that the superconductor is second kind. This same problem has been approached by looking at the Abrikosov flux tubes generated by propagating charges. The idea is to compare the dependence of the electric field inside the tube on the transverse distance $x_\perp$ from the center of the tube, with what is expected from London equations. Their result is that the system seems to be is at the border between first and second kind [18]. The method is ingenuos. However derivatives are approximated by finite differences, the penetration depth being a few lattice spacings (2-3), and this can produce systematic errors. Our method would give a more precise determination if we were able to determine better the mass M of Eq.(45). The question deserves further study.

[1] In ref. [4] the field was called magnetic.

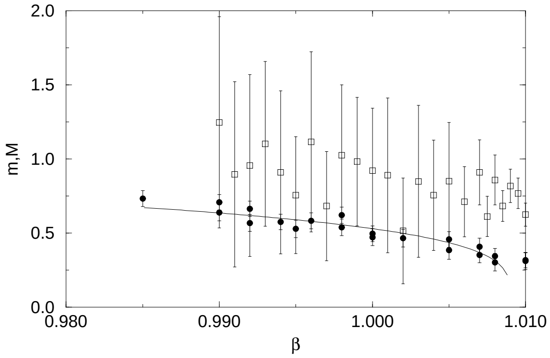

Fig.5 Mass of the monopole M (squares), and mass of the dual photon m (circles) vs. β.

Finally we want to comment on the possibility of determining numerically $\langle\mu\rangle$ directly and not through the measurement of ρ. As we have seen this is not strictly necessary, since ρ gives complete information about the phase transition. However the problem has some interest by itself.

The definition of $\langle\mu\rangle$ is $\langle e^{\beta S'}\rangle$, the average being performed with the weight $\mathcal{D}\theta e^{\beta S}/Z$. S' is itself a random variable in this ensemble which has some average value $\langle S'\rangle$ with a width $\sigma = \sqrt{\langle S'^2\rangle - \langle S'\rangle^2}$.

A general theorem of probability theory states that if a random variable is distributed with a probability law $p(x)$, with $\int p(x)\mathrm{d}x = 1$, then its average $x_n = \frac{1}{n}\sum_k x_k$ is distributed as a gaussian for large n if and only if [19]

$$\lim_{X\to\infty} \frac{X^2 \displaystyle\int_{|x|>X} p(x)\mathrm{d}x}{\displaystyle\int_{|x|<X} x^2 p(x)\mathrm{d}x} = 0 \tag{59}$$

If Eq.(59) holds, then

$$\langle x_n\rangle \underset{n\to\infty}{\to} \langle x\rangle = \int x\, p(x)\mathrm{d}x \tag{60}$$

and the width of the distribution is in this limit

$$\sigma_n = \frac{\sigma}{\sqrt{n}}$$

with

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$$

If we denote by y the variable $\beta S' - \langle\beta S'\rangle$ and by $\pi(y)$ its probability distribution, then the variable μ

$$\mu = \exp(\beta S') = \bar{\mu} exp(y) \qquad (\bar{\mu} = exp(\langle\beta S'\rangle))$$

will be distributed as

$$p(\mu) = \pi\left(\ln\left(\frac{\mu}{\bar{\mu}}\right)\right)\mathrm{d}\ln\left(\frac{\mu}{\bar{\mu}}\right) \tag{61}$$

If π decreases as $\exp(-y^2/2\sigma_y^2)$ as $y\to\infty$ then the probability distribution (61) obeys the hypotesis (59) of the theorem of central limit. In fact a much slower decrease would be enough.

If, for the sake of the argument, we assume that $\pi(y)$ is gaussian, then we easily compute, by use of Eq.(61)

$$\langle\mu\rangle = \bar{\mu}\exp(\frac{\sigma_y^2}{2}) \tag{62}$$
$$\sigma_\mu = \bar{\mu}\exp(\sigma_y^2)$$

Eq.'s(62) show why a direct determination of $\langle\mu\rangle$ is affected by wild fluctuations: the width is indeed bigger than the value of $\langle\mu\rangle$ itself. The exponential dependence on S' strongly distorts the distribution when going from S' to μ.

The histogram of the values of μ is related to the constrained potential by the relation [10,20]

$$\exp\left(-V(\Phi)\right) = \int[\mathcal{D}\theta]\exp(\beta S)\delta(\mu-\Phi)$$

$V(\Phi)$ has a minimum at $\langle\beta S'\rangle + \frac{\sigma_y^2}{2}$.

If instead we construct the histogram of $\beta S'$ itself, the minimum will appear at $\langle\beta S'\rangle$ which is displaced by $\frac{\sigma_y^2}{2}$ with respect to the real minimum.

The problem is that the histogram in μ is exponentially large to fill adequately, since μ fluctuates on an exponential scale (typical values of μ on a configuration for a reasonable lattice size range from 10^{150} to 0). A histogram of $\log\mu$, i.e. of $\beta S'$ is easier to compute.

However to go back to the distribution in μ, i.e. to compute $\langle\mu\rangle$ and σ_μ, we must know the distribution $\pi(y)$ with great precision. In the gaussian approximation the solution is given by Eq.(62). A cluster expansion can be attempted, to evaluate non gaussian effects, but the problem is only shifted. Higher cumulants of $\pi(y)$ are more and more noisy to determine numerically, and the computer time needed becomes comparable to the one needed for the direct determination of $\langle\mu\rangle$.

Finally a finite size scaling analysis would be needed, analogous to what we did in sect.4.

This is to justify why we used ρ to extract information on the phase transition, instead of $\langle\mu\rangle$ itself, or of its effective potential.

The problem is currently under further study.

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